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Technical Note TN 16

A note on a paper by H. Levine

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March 1960

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Summary

The diffusion problem considered by H. Levine in a previous paper is solved here in a different way by making use of the known solution of the corresponding problem of Green.

Introduction

In a recent paper H. Levine [1)] considers the problem of solving a non-homogeneous Helmholtz equation in an angle. With a slight change of notation the problem can be formulated as follows. To find a solution $\psi(r,\varphi)$ in the angle $0<\varphi<\theta$, $0<r<\infty$ of

$$(\Delta-1)\psi = -1 \tag{1.1}$$

for which

$$\psi = 0 \quad \text{at} \quad \varphi = 0 \quad \text{and} \quad \varphi = \theta \quad . \tag{1.2}$$

The corresponding problem of Green where $G(r,\varphi,r_0,\varphi_0)$ satisfies

$$(\Delta-1)G = -r_0^{-1}\,\delta(r-r_0)\delta(\varphi-\varphi_0) \tag{1.3}$$

and

$$G = 0 \quad \text{at} \quad \varphi=0 \quad \text{and} \quad \varphi=\theta \tag{1.4}$$

has been solved in Lauwerier [2)]. Then the solution of (1.1) and (1.2) is simply

$$\psi(r,\theta) = \int_0^\infty r_0 dr_0 \int_0^\theta G(r,\varphi,r_0,\varphi_0)d\varphi_0. \tag{1.5}$$

It will be shown that from (1.5) the solution as given by Levine can easily be derived. We have explicitly

$$G(r,\varphi,r_0,\varphi_0) = \begin{cases} 2\theta^{-1}\sum_{m=1}^{\infty} K_{m\nu}(r)\, I_{m\nu}(r_0)\sin m\nu\varphi \;\sin m\nu\,\varphi_0 & \text{for } r>r_0 \\ 2\theta^{-1}\sum_{m=1}^{\infty} I_{m\nu}(r)\, K_{m\nu}(r_0)\sin m\nu\varphi \;\sin m\nu\,\varphi_0 & \text{for } r<r_0, \end{cases} \tag{1.6}$$

with $\nu=\pi/\theta$, so that by means of (1.5)

$$\psi(r,\varphi) = \frac{4}{\pi}\sum_1 \frac{\sin m\nu\varphi}{m} \; L_{m\nu}(r), \tag{1.7}$$

where the auxiliary function $L_\mu(x)$ is defined by

$$L_\mu(x) = I_\mu(x) \int_x^\infty t\, K_\mu(t)dt + K_\mu(x) \int_0^x t\, I_\mu(t)dt, \tag{1.8}$$

and where $\sum_1$ indicates summation over odd numbers m=1,3,5...

A few properties of $L_\mu(x)$ which may be considered as a modified Lommel function (cf. Watson [3]) will be given in the next section. We shall prove that

$$L_\mu(x) = 1 - \mu \int_0^\infty e^{-\mu u} \cos(x\ \mathrm{sh} u)du. \tag{1.9}$$

If this is substituted in (1.7) it follows that

$$\psi(r,\varphi) = 1 - 4\theta^{-1} \sum_1 \sin m\nu\varphi \int_0^\infty e^{-m\nu u} \cos(r\ \mathrm{sh} u)du$$

and hence

$$\psi(r,\varphi) = 1 - 2\theta^{-1}\ \mathrm{Im} \int_0^\infty \frac{\cos(r\mathrm{sh}u)}{\mathrm{sh}\nu(u+i\varphi)}\, du, \tag{1.10}$$

which is equivalent to the result obtained by Levine (l.c. formula (2.18)).

§ 2. Properties of $L_\mu(x)$

It can easily be verified that $L_\mu(x)$ is a solution of the non-homogeneous Bessel equation

$$L_\mu'' + \frac{1}{x} L_\mu' - \left(1+\frac{\mu^2}{x^2}\right)L_\mu = -1 \tag{2.1}$$

and that for $\mu > 0$

$$L_\mu(0) = 0, \quad L_\mu(\infty) = 1. \tag{2.2}$$

By applying cosine transformation in the form

$$L_\mu(x) = 1 - \int_0^\infty \cos xt\ f(t)dt \tag{2.3}$$

the equation (2.1) changes into

$$(1+t^2)f'' + 3tf' + (1-\mu^2)f = 0 \tag{2.4}$$

with $f'(0) = -\mu^2$.

By making the substitution $t=\mathrm{sh}u$ the equation (2.4) becomes

$$\frac{d^2}{du^2}(\mathrm{ch}u\ f) = \mu^2(\mathrm{ch}u\ f) \tag{2.5}$$

with $f'(0) = -\mu^2$.

This has the solution vanishing at infinity

$$f(u) = \mu e^{-\mu u}/\mathrm{ch}u \quad . \qquad (2.6)$$

Therefore (2.3) becomes

$$L_\mu(x) = 1 - \mu \int_0^\infty e^{-\mu u} \cos(x \,\mathrm{sh} u) du. \qquad (2.7)$$

A series valid for small values of x can easily be derived. If $\mu \neq 2,4,6\ldots$ we obtain (cf. Watson l.c.)

$$L_\mu(x) = - \sum_{m=1}^{\infty} \{(4-\mu^2)(16-\mu^2)\ldots(4m^2-\mu^2)\}^{-1} x^{2m}.$$

If μ is a positive even integer logarithmic terms are obtained. The first term is, however, always $(\mu^2-4)^{-1}x^2$ except for $\mu=2$ when it is $-\frac{1}{4} x^2 \ln x$.

The asymptotic expansion for large values of x is (cf. Watson l.c.)

$$L_\mu(x) \approx 1 - \sum_{m=0}^{\infty} \mu^2(4-\mu^2)\ldots(4m^2-\mu^2)x^{-2m-2}.$$

If $\mu=2,4,6\ldots$ the series on the right-hand side breaks off but then we have the exact relation

$$L_{2n}(x) = \sum_{m=0}^{n} (-1)^m n^2(n^2-1)\ldots(n^2-(m-1)^2)(\tfrac{1}{2}x)^{-2m} - (-1)^n 2nK_{2n}(x).$$

References

1) Levine, H., Appl. Sci. Res. B 8 (1960) 105-127.

2) Lauwerier, H.A., Proc.Kon.Ned.Ak.v.Wet. Amsterdam. Proc.A, 62, no 5, 475-488.

3) Watson, G.N., Theory of Bessel functions. Cambridge (1944) 10.7.